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Statistical-Economic Design of Control Chart If The Vector of Target Values of Multi-Variables Shifts With Time

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ABSTRACT

In this paper, the statistical-economic design of a multivariate special triangle control chart is proposed to control the processes of the quality characteristics or financial indices shifting with time. A multi-objective programming with several constraints is used to determine optimal solutions of control region (probability of false alarm), the power of finding out assignable cause(s), sample interval and sample size. An application of the statistical-economic design of a Multivariate special triangle control chart is illustrated to control the soundness of insurers of U.S

KEYWORDS

Multivariable; Special triangle control chart; Optimization; Shifting with time; Soundness of insurers

INTRODUCTION

Mao (2019 a) presents a new quality control technique. The optimization adjustment interval of one dimensional quality characteristic are extended to multi-dimensional case in which the vector of the quality characteristics or important financial indices of firms shifts with time due to seasonal changes or cyclical change of economic environment, and also there are time-varying correlated among each other. The statistical-economic design of a multivariate special triangle control chart is proposed to control such kinds of processes. An application is illustrated to monitor the soundness of insurers of U.S. Optimal solutions of control region, power of finding out assignable cause(s), sample size and sample interval are found out.

In business process, the process means shift gradually with time. For instance, the means of some performance type financial indices gradually increases or some consumption type financial indices gradually decrease And we must distinguish three different situations. One is that the shift of the means is resulting from the normal growth and development of firms and another is that it is due to the unnormal events happening in firms, such as, the underwriting premium or liabilities suddenly increases in a very short time to very high level and even excesses the normal underwriting capability. Since the quality of underwritten insurance policies is not good, it will cause very big insolvency risk to insurance companies. The third situation is that the process is affected by some causes and the financial indices shift gradually from the targets due to seasonal changes or cyclical change of economic environment. And these changes are unavoidable within an allowable range. The important thing for insurance companies is to use suitable control techniques to dynamically control the means of important financial indices under the target level, find and correct unnormal situations as long as the control chart alarms the signal of out of control. It is obvious different from production process where in any situation that the target means is always a constant even though the quality characteristics change with time due to tool wearing or some other reasons such as material consumption. This is because the engineering specification limit or region is fixed. The control target is to keep the process means at the target value presetting by adjustment the process cyclically. Mandal (1969) (also see Montgomery (2009)) suggests using "Trends $\overline{X}-R$ control charts" to jointly control the mean and the deviation of a process. The fitted regression equation is used as the central control limit with the parallel upper and lower control limits. The width of the control limits is 6σ. The process needs to be adjusted as long as the regression line is over the presetting maximum level of process mean. Quensenberry (1988), Wu (1998), Cheng and Fricker Jr. (1999) and Kang et al.(1999) also address the quality control of such kinds of production processes. Spiring (1991) proposes a method to evaluate the process capacity when there exists an unavoidable systemic cause. Spiring and Cheng (1998) propose a single variable control chart, the MSE chart, to monitor the location and the scale simultaneously. Mao (1995 a) presents a new technique of multi-variate joint quality control. It can be used to monitor the change of mean vector and covariance simultaneously. Its application is illustrated with an example. Finally, the characteristic of Average Run Length (ARL) is discussed. Chao and Cheng (1996) discuss the semi-circle control chart to control both location and variation of quality characteristic simultaneously. Mao et al (2014) apply dynamic monitoring to control and predict insurers' financial strength. They present a new statistic that combines means, variances, and co-variances of the multivariate financial indices as a new type of control tool. They use data from the U.S. property and casualty insurers from 2001 through 2010 to determine the control regions and provide two examples to illustrate the application of their proposed methodology. Mao (1997) discuss the control and optimization of adjustment interval when one dimensional quality characteristics shifts with time. Mao and Cheng (2016) extend it to use joint trends semi-circle control chart to control this type of processes effectively. An optimization model is suggested to determine the optimal interval of adjustment. They also discuss the average run length of the proposed control chart and the extension to the EWMA chart. An example is used to illustrate its application in a production process. There are lots of literature on economic or statistical and economic design of multi-variable processes (For the review of them, please see Mao (2019 a,b).

There are some papers approaching the economic or statistical-economic design based on Taguchi (1986), for example: Krishnamoorthi et al. (2009) Alexander et al.(1995). Cai et al. (2002) present the economic design of control chart for trended processes. They address appropriate timing problem of making adjustment of trended process through economic design of its control chart. However, we believe that there are some limitations on this study. Taking the shift of quality characteristic from the acceptable level as only assignable cause to be determined and corrected may not be best strategy. Not considering other important and possible assignable cause(s) and corresponding cost will cause additional control cost and result in ineffective process control. The economic design or statistical-economic design of control chart generally focus on optimization of three important parameters of the upper and lower limits of control charts, sample interval and sample size.

In this article, we establish multi-variable control chart and we assume that the vector of multi-financial indices change with time. We apply three dimensional and multivariate special triangle control chart to monitor process mean vector shifting with time and covariance jointly. We also discuss the statistical- economic design of the control chart we present. Different from Cai et al. (2002), which considers the means of quality characteristic expressing unacceptable means as out of control state, we divide assignable causes into two kinds, that is, avoidable and unavoidable cause(s). Our optimization problems is to establish economic design model. It only takes into account of avoidable causes to determine optimal control limit (region) and sample interval. The total cost of process control cost includes the control cost of production and the loss of consumer due to out of control. It should be especially noticed that our approach in this article has obviously three aspects of importance. First one is its flexibility and universal adaptability，that is, it has universal and widespread application value, which can be applied in the process control of almost all industries. Second one is its simplicity and effectiveness. Third one is its dynamic nature, that is, it can be used in tracing and monitoring of the process in which the vector of target values of multi-variables changes with time, which is a widespread phenomenon in the growth type of industries. In fact, the target values of indices reflecting the growth of the firms gradually increases with time but those reflecting their consumption reduce with time. Figure 1 and Figure 2 displays the time series patterns of the means of main financial indices of an insurance company in U.S. Figure 3 describe the change patterns of the means of asset and liability values of a main bank on U.S.

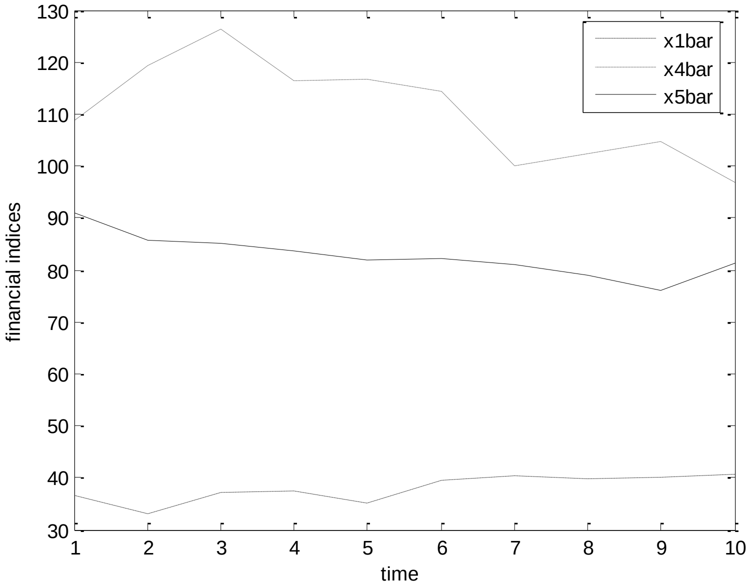

Figure 1. The change pattern of the mean of financial indices of insurers (1)

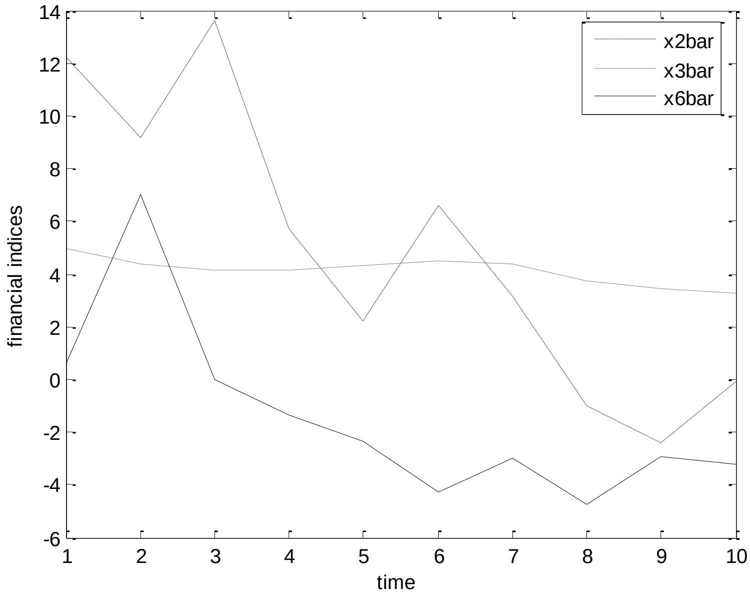

Figure 2. The change pattern of the mean of financial indices of insurers

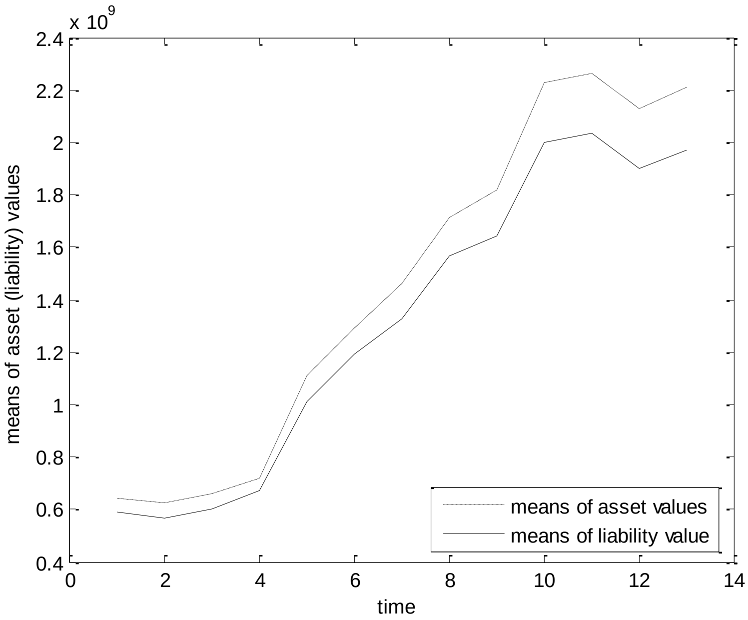

Figure 3. The change patterns of the values of assets and liabilities of a bank

RESEARCH METHODS

List of notations used in the model

p : the dimension of multi-variables;

$\boldsymbol{X}(t)$: the vector of the values of multi-variables concerned and t the time;

$\boldsymbol{m}_t$: the vector of the target values of multi-variables at time t;

$\boldsymbol{b}$: the vector of the speed of multi-mean vector shifting from the vector of the target values of multivariable;

Σ_0 : the covariance matrix of the multi-variables;

σ_{ij} : the covariance between variables i and j if $i \neq j$ and it is the variance of variable i if $i = j$, $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, k$.

$LA(t)$: the measurement of random variation of j-th variables at time t, $j = 1, 2, \cdots, k$.;

$LB_j(t)$: the measurement of the un-avoidable assignable cause at time t, $j = 1, 2, \cdots, k$;

$\overline{X}_j(t)$: the vector of $j-$th sample means of multi-variables at time t, $j = 1, 2, \cdots, k$.

Model for monitoring the vector of target means of multi-variables shifted with Time

Similar to Mao (2019 (a)), we assume that there are p variable for a firm. Let $\boldsymbol{X}(t)$ be the vector of the values of the characteristics that changes with time. We use the linear equation to describe this change. let $\boldsymbol{X}(t)$ be a multivariable continuous function of t, and $\boldsymbol{X}(t) = \boldsymbol{m}_t + \boldsymbol{b}t + \boldsymbol{\varepsilon}$. $\boldsymbol{X}(t) \sim MN(\boldsymbol{m}_t + \boldsymbol{b}t, \Sigma_0)$, where m_t is the vector of the target mean at time t, $\boldsymbol{b}$ the vector of the speed of multi-mean vector shifting from the vector of the target values of multivariables and t the time. For the methods of determining parameter vector of $\boldsymbol{b}$, please refer to Mao (2019(a)). Different from Mao (2019 (a)), we assume that the vector

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of target means m_t is a function of time t. One typical example is dynamic monitoring of multi-financial indices.

Similar to Mao (2019, a), we define $\boldsymbol{x}_{pt} = \begin{pmatrix} x_{11t} & x_{12t} & \cdots & x_{1nt} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(p-1)1t} & x_{(p-1)2t} & \cdots & x_{(p-1)nt} \\ x_{p1t} & x_{p2t} & \cdots & x_{pnt} \end{pmatrix}, t = 1, 2, \ldots T$, to be the t random sample of size n taken at time t from a given process. The tth sample mean vector, $\bar{\boldsymbol{X}}_t = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i, t = 1, 2, \ldots, T$, follows a multi-variable normal distribution, that is, $\bar{\boldsymbol{X}}_t(t) \sim MN_p(\boldsymbol{m}_t + \boldsymbol{b}t, \Sigma_0 / n), t = 1, 2, \cdots, T.$, where Σ_0 is the covariance matrix of the multi-variables expressed as:

$$\Sigma_0 = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \vdots & \vdots & \sigma_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{p1} & \cdots & \cdots & \sigma_{pp} \end{pmatrix}$$

Assume that the expected vector of observations vector $\boldsymbol{x}_{ij}$ belonging to same sample change with time, that is, the expected value vector of the i th value of the j th sample is

$$E(\boldsymbol{x}_{it}) = \hat{\boldsymbol{x}}_{it} = \boldsymbol{m}_t + \boldsymbol{b}t \quad . \qquad (1)$$

The square sum of deviations between the i th observation and the mean of j th sample could be expressed as:

$$\sum_{i=1}^{n} (\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_t)'\Sigma_0^{-1}(\boldsymbol{x}_{it} - \bar{\boldsymbol{x}}_t)$$
$$= \sum_{i=1}^{n} (\boldsymbol{x}_{it} - \hat{\boldsymbol{x}}_{it})'\Sigma_0^{-1}\left(\boldsymbol{x}_{it} - \hat{\boldsymbol{x}}_{it}\right) + \sum_{i=1}^{n} (\hat{\boldsymbol{x}}_{it} - \bar{\boldsymbol{x}}_t)'\Sigma_0^{-1}(\hat{\boldsymbol{x}}_{it} - \bar{\boldsymbol{x}}_t), \qquad (2)$$
$$= LA(t) + LB(t), t = 1, 2, \cdots, T$$

where $LA(t)$ is the measurement of random variation of multivariables at time t, $LB(t)$ is the measurement of the un-avoidable assignable cause at time t, $0 \le t \le T$ and T is control period. We assume that the un-avoidable assignable cause is independent of other possible causes and both of them are functions of time, where

$$LB(t) = \sum_{i=1}^{n} (\hat{\boldsymbol{x}}_{it} - \bar{\boldsymbol{x}}_t)'\Sigma_0^{-1}\left(\hat{\boldsymbol{x}}_{it} - \bar{\boldsymbol{x}}_t\right)$$
$$= \sum_{i=1}^{n}\left(\boldsymbol{m}_t + \boldsymbol{b}t - \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{m}_t + \boldsymbol{b}t + \boldsymbol{\varepsilon}_{it})\right)'\Sigma_0^{-1}\left(\boldsymbol{m}_t + \boldsymbol{b}t - \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{m}_t + \boldsymbol{b}t + \boldsymbol{\varepsilon}_{it})\right)$$
$$= \sum_{i=1}^{n}\left(\frac{n-1}{n}\boldsymbol{b}t + \boldsymbol{\varepsilon}_{it}\right)'\Sigma_0^{-1}\left(\frac{n-1}{n}\boldsymbol{b}t + \boldsymbol{\varepsilon}_{it}\right)$$

$$t = 1, 2, ..., T \tag{3}$$

The expected values of $LB(t)$ is

$$E(LB(t)) = E\sum_{i=1}^{n}\left((n-1)\boldsymbol{b}t + \boldsymbol{\varepsilon}_{it}\right)'\Sigma_0^{-1}\left((n-1)\boldsymbol{b}t + \boldsymbol{\varepsilon}_{it}\right)$$
$$= \sum_{i=1}^{n}\left((n-1)\boldsymbol{b}t\right)'\Sigma_0^{-1}\left((n-1)\boldsymbol{b}t\right) + n$$
$$t = 1, 2, ..., T \quad . \tag{4}$$

The proof of independnece of $LA(t)$ and $LB(t)$ is almost same as that in Mao (2019, a). Please see Appendix 1.

Similar to Mao (2019 (a)), we let

$$LS_a^{2*}(t) = \frac{\sum_{i=1}^{n}(\boldsymbol{x}_{it} - \hat{\boldsymbol{x}}_{it})\Sigma_0^{-1}(\boldsymbol{x}_{it} - \hat{\boldsymbol{x}}_{it})}{n-2}, \tag{5}$$

$$LW^V(t) = (\overline{\boldsymbol{x}}_t - \hat{\boldsymbol{x}}_t)'\Sigma_0^{-1}(\overline{\boldsymbol{x}}_t - \hat{\boldsymbol{x}}_t), \quad LZ_a^V(t) = \frac{n-2}{n}LS_{a(V)}^{2*}(t) \tag{6}$$

and $LT_a^V(t) = LW^T(t) + LZ_a^V(t)$ (7)

By Hohollon theorem (2013) : if $y \sim N(\mu, \sigma^2)$ and iid, then $\sum_{i=1}^{m}\left(\frac{y-\mu}{\sigma}\right)^2 \sim \chi^2(n)$. It is easily know that

$$LW^V \sim \frac{1}{n}\chi_p^2. \tag{8}$$

$$LS_{a(V)}^{2*} \sim \frac{1}{n-2}\chi_{(n-2)p}^2, \tag{9}$$

and $LT_a^V \sim \frac{1}{n}\chi_{(n-1)p}^2$. (10)

We apply $LT_a^V(t)$ to construct a " Dynamic Multivariable Triangle Control Chart ", the $LT_a^V(t)$ Chart, to control both the location and the variation of the process.

Since $P(LT_a^V \le LV) = P\left(nLT_a^V \le nLV\right) = P\left(LT_a^V \le \frac{1}{n}\chi_{(n-1)p,1-\alpha}^2\right) = 1-\alpha$, （11）

We have the intercept $LV = \frac{1}{n}\chi_{(n-1)p,(1-\alpha)}^2$, (12)

where $\chi_{(n-1)p,(1-\alpha)}^2$ is the $100(1-\alpha)\%$ percentile of $\chi_{(n-1)p}^2$. Since the values of r^{LT} are same as those in Mao (2019 (a)) Table1, where the author gives the values of the radius with

$\alpha = 1-(1-0.0027)^{p}$, n = 2(1)20, $p = 1(1)5$. We do not want to repeat it. The readers with interests can refer to Mao (2019 (a)).

Statistical-Economic Design of Dynamic Multivariable Triangle Control Chart

Regarding to economic or statistical-economic design of control charts, most of literature is focused on $\overline{X}$ control chart and only very limited literature consider the joint control charts, such as $\overline{X}-R$ control chart. Since it is important to design process control procedure by taking into account of both means and dispersal of variables simultaneously, in following, we will establish model of statistical-economic model of dynamic multivariable special; triangle control chart, which can be used to jointly control the vector of means and covariance of multivariable under consideration of the vector of target means of multivariable shifting with time.

Multi-objective economic statistical design is first introduced by Evans and Emberton (1991) for joint X bar and R control charts. In their studies, multi-objectives, including cost function and statistical properties are optimized simultaneously. Therefore, optimal design of a control chart is represented as a Multiple Criteria Decision- Making (MCDM) problem. Safaei (2012) et al. present a multi-objective model of the economic statistical design of the X-bar control chart, which incorporating the Taguchi loss function and the intangible external costs. The model minimize the expected hourly loss cost while minimizing out-of-control average run length and maintaining reasonable in-control average run length. The Pareto optimal solutions are obtained by evolutionary algorithm, namely NSGA-II. Bahiri et al. (2013) propose economic-statistical design of $\overline{X}$ control chart by using multi-objective genetic algorithm. They consider off the trade between average time needed to detect out of control (ATS) and the control cost taken in whole control period in their multi-objective function.

In our model, we also apply multi-objective function. According to Taqucis (1986), any shift from the target value represents the loss. Different from Taqucis (1986), we consider the social loss resulting from both the shift from the target values and un-normal increase of covariance in our design. Similar to Bahiri (2013), we also consider off the trade between total expected social loss and ATS. In the following, we will discuss the case of optimal control of multi-financial indices

In the study of economic design of control charts applied in the control of production process, there are three important parameters to be determined. They are sample size, sample interval and control limits (region), while the selection of optimal sample size generally only consider the effect of sampling cost on total control cost. However, one important fact in monitoring financial performance of firms is that sample cost is much less important than other cost. In our optimal design schedule discussed in this sub-section, we discuss the optimal design of multivariate special triangle control chart from aspect of statistical and economics and we neglect sample cost.

Therefore, our optimal design of the control chart for monitoring financial state of a firm is as to determine three parameters including control bound (region), sampling interval and and sampling size. A great difference between the control of production process and financial performance or insolcency of firm is that the loss resulting from any missed alarms will be much greater than that resuting from false alarms (Mao and Hao (2019), because any alarms of out of control of firms's financial indices indicates the great risk of bankruptcy of firms and the loss will be fatal if it really happens. While it is very difficult to estimate the loss caused by missed alarm accurately, in the establishment of optimization model, we minimize the total expected control cost, at the same time, to assure the power of the control chart is greatest and average time needed to detect out of control is minimized. Therefore, it is an optimization problem of multi-objectives.

Assumptions

(1) Assume that the dynamic statistic of p dimensional quality variables, $LT_a^T(t) \sim \chi^2_{(n-1)p}$, where n is sample size, The process may be affected by $v(v \geq 1)$ kinds of assignable causes which is avoidable, and the non-centrality parameter would change from 0 to δ_k, $k = 1, 2, \ldots, v$. when the process is out of control as a result of the occurrence of the assignable cause being avoidable, where non-centrality parameter $\delta_k = \frac{1}{n}(\boldsymbol{U}_k - \boldsymbol{U}_0)' \Sigma_k^{-1} (\boldsymbol{U}_k - \boldsymbol{U}_0), k = 1, 2, \ldots, v$, (13)

Where $\boldsymbol{U}_k$ and Σ_k are the vector of the means and covariance when the process is affected by $k-th$ assignable cause being avoidable. The process will not affected by other assignable causes which is avoidable when it is affectd by one assignable cause being avoidable. It is also assumed that if the process is out of control it remains in that state until detected and corrective action takes.

(2) Assume that the assignable cause occurs during a time interval following a Poisson process, that is, the occurrence of the assignable cause are independent exponential variables with means of $1/\lambda_k, k = 1, 2, \ldots .v$.

(3) Assume that the process is not shut down during the search for the assignable cause. It means that the business of the firm will not be interrpted when the out of control alarm is sounded.

A_0: the expected direct and indirect loss to the firm bacause of out of control of the firm including the reduction of the profit or the value of shareholders'and the loss resulting from the reduction of the demand.

The models of the the cycle of the process and the costs

Since the sampling cost is much smaller comparing with other costs, we neglect it.

Similar to Mao (1995), the cycle of the process control is

$$T = \frac{1}{\lambda} + \frac{Dq(t)}{e^{\lambda h(t)} - 1} + \sum_{k=1}^{v} \left(\frac{\lambda_k h(t)}{P_k(t)} - \lambda_k T_k \right) / \lambda + D + \sum_{k=1}^{v} \lambda_k \tau_k / \lambda \quad , \qquad (14)$$

where D is the expected search time for real or false alarms, τ_k is the time of correcting, T_k is the average time of occurrence of the k^{th} assignable cause being avoidable within a sampling interval $h(t)$ if it occurs between s^{th} and $(s+1)^{th}$

interval, $$T_k = \frac{\int_{sh(t)}^{(s+1)h(t)} (t_0 - sh(t)) \lambda_k e^{-\lambda_k t_0} dt_0}{\int_{sh(t)}^{(s+1)h(t)} \lambda_k e^{-\lambda_k t_0} dt_0} = \frac{1}{\lambda_k} - \frac{h(t)}{\exp(\lambda_k h(t)) - 1}, \qquad (15)$$

k^{th} assignable cause being avoidable when it happens, p_k is the power of control chart and

$$p_k = P\left(LT_a^V \geq \frac{1}{n}\chi^2_{(n-1)p,\delta_k,1-q(t)} \right), \quad (16)$$

and $q(t)$ is the probability of false alarming, and $q(t) = P\left(LT_a^V \geq \frac{1}{n}\chi^2_{(n-1)p,1-q(t)} \right)$. (17)

Different from Cheng and Mao (2011), the total expected cost does not include the expected cost incurred due to a higher rate of defectives when the process is out of control, the situation of the control of production process, and also the total expected cost is time-varying. Hence, the total expected control cost can be written as

$$E(C_1(t)) = Da_1 \cdot \frac{q(t)}{e^{\lambda \cdot h(t)} - 1} + a_1 D + \sum_{k=1}^{v} (\lambda_k / \lambda)\tau_k a_2(k)$$
$$+ \sum_{k=1}^{v} (\lambda_k / \lambda) l_k \left(|\Sigma_k| - |\Sigma_0| (\boldsymbol{U}_0 - \boldsymbol{m}_t)'(\boldsymbol{U}_0 - \boldsymbol{m}_t) \text{-} (\boldsymbol{U}_k - \boldsymbol{m}_t)'(\boldsymbol{U}_k - \boldsymbol{m}_t) \right) B_k(t), \ 0 \leq t \leq in(T),$$

(18)

where a_1 is the average cost per unit time of searching for the assignable cause being avoidable, $a_2(k)$ is the avarage cost per unit time of adjusting and dealing with k^{th} assignable cause being avoidable and

$$B_k(t) = h(t) / p_k - T_k + D, \quad (19)$$

A_0 is the expected direct and indirect loss to the firm when the business of the firm is out of control and l is its expected direct and indirect loss to the firm per unit of the square errors because of out of control of the firm, and

$$l = \frac{A_0}{\left| \left(\sum_{k=1}^{v} \boldsymbol{U}_k (\lambda_k / \lambda) - \boldsymbol{U}_0 \right)' \left(\sum_{k=1}^{v} \boldsymbol{U}_k (\lambda_k / \lambda) - \boldsymbol{U}_0 \right) \right|}. \quad (20)$$

The multi-objective function is expressed as following:

$$\begin{aligned} &\text{Min } E(C_1(t)) \\ &\text{st.} \min(p_k, k = 1, 2, \ldots, v) \geq G_1 \\ &\quad \max(ATS_k(t), k = 1, 2, \ldots, v) \leq G_2(t), \\ &\quad q(t) \leq G_3(t). \end{aligned} \quad (21)$$

where ATS_k is average time needed to detect out of control due to k^{th} assignable cause being avoidable, $ATS_k(t) = \frac{h(t)}{p_k}$ (Bahiri et al., 2013). (22)

and G_1 $LV^*(t)$ and $G_2(t)$ are given constants. We can find optimal values of parameters of sampling interval, $h(t)$, sampling size $n(t)$ and the intercept of triangle control region through grid search method.

RESULTS AND DISCUSSION

Statistic and Economic Design by Using of the data of insurers of U.S

We begin by discussing statistical and economic estimation of the optima control region, sample interval and the vector of target value of unified statistic using historical (2001 through 2010) data from all U.S. property/casualty (P/C) insurance companies in financially strong condition, as provided by SNL Financial[1].

We select six key financial indices according to their impact on the soundness of insurers. Appendix 2 lists these six key indices and their detail descriptions. We obtain co-variances matrix $\Sigma_{0(\bar{x})}$ by using the historical data as:

$$\Sigma_{0(\bar{x})} = \begin{pmatrix} 4.346 & -9.593 & -0.274 & -31.194 & -0.934 & -7.397 \\ -9.593 & 33.871 & 1.202 & 93.564 & 4.119 & 20.476 \\ -0.274 & 1.202 & 0.117 & 3.028 & 0.076 & 0.633 \\ -31.194 & 93.564 & 3.028 & 324.483 & 10.961 & 66.109 \\ -0.934 & 4.119 & 0.076 & 10.961 & 0.696 & 2.443 \\ -7.397 & 20.476 & 0.633 & 66.109 & 2.443 & 15.243 \end{pmatrix}$$

Table 2. shows the results of calculations of statistic $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6, LW^V(t)$ and by using the data of five top property-casualty insurance companies in the United States (SNL Financial, 2011). The companies are State Farm Mutual Automobile Insurance Company, Allstate Insurance Company, Nationwide Mutual Insurance Company, Travelers Commercial Insurance Company and Travelers Casualty and Surety Company of America.

Table 1. Calculations of $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6$ for 2001-2010 using the data of five top P/C insurance companies in the U.S.

Number	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$	$\bar{x}_5$	$\bar{x}_6$	LW_a^T	LS_a^T
1	36.698	12.232	4.988	108.710	90.892	0.622	0.702	1.374
2	33.082	9.160	4.376	119.378	85.668	6.990	0.334	1.503
3	37.022	13.608	4.142	126.340	85.172	-0.014	0.190	1.753
4	37.458	5.736	4.118	116.338	83.668	-1.356	0.261	2.356
5	35.226	2.228	4.302	116.558	81.934	-2.338	0.394	3.091
6	39.594	6.612	4.492	114.340	82.060	-4.300	0.314	3.327
7	40.274	3.168	4.358	100.024	80.904	-2.980	0.331	4.452
8	39.656	-0.980	3.740	102.480	78.976	-4.732	0.292	6.200
9	40.094	-2.388	3.436	104.790	76.052	-2.914	0.132	6.604
10	40.640	-0.088	3.242	96.900	81.434	-3.208	0.854	2.921
$\bar{\bar{x}}_i$	37.974	4.929	4.119	110.586	82.676	-1.423		

Table 2. The Results of linear regression

jth-index	b_j	std-Err.	q-quantile
1	0.6700	0.1723	0.0046
2	-1.6241	0.2925	0.00054
3	-0.1418	0.0342	0.0032
4	-2.2962	0.7538	0.0159

[1] SNL Financial LC, Online database at http://www.snl.com, accessed in May 2011, Charlottesville, Virginia.

5	-1.1611	0.2323	0.0011
6	-0.8135	0.2759	0.0185

Other important parameters are as follows:

$U_0 = [37.974, 4.924, 4.119, 110.586, 82, 676, -1.423]$,

$U_1 = [37.974, 4, 924, 4.119, 110.586, 82, 676, -1.423] \times (1-10\%)$,

$k = 1, a_1 = 2, a_2 = 130, D = 3, \lambda = 0.17, \tau = 0.07, G_1 = 0.98$,

$\boldsymbol{m} = [36.861, 5.610, 4.115, 111.62, 83.4790, \text{-}1.290], g = 0.3$

$A_0 = 1.9, G_2 \le 0.7.$

where U_0 is the vector of means of financial indices when the process is in control and U_1 is the vector of means when the process is out of control. By using the data given in Table 1,Table 2 and other data given above, we can get optimal soluions of control region,sample interval, sample size and the power to find out assignable cause(s).

To simplify calculation, we assume only the vector of financial indices change, but the volatility and covariance do not change. We also assume that the process is only possiblally affected by one assignable cause which results in the process out of control. With the help of Math lab and multi-objective programming, we can obtain optimal solutions of sample interval, control region, probability of false alarm, power of finding out assignable cause(s), average time of finding assignable cause(s) based on the criteria of statistical-economic design of control chart.

Table 3 list optimal solutions of statistical-economic design of multivariate special triangle control chart. Figure 4 displays the multivariate special triangle control chart. The correct selection of the vector of target values is very important for monitoring and improving the financial performance of a firm. Therefore, the process manager must inspect the process characteristics and adjust them periodically. Well known expert of quality management, Dr. Deming emphasizes the importance of quality improvement. Setting the vector of financial indices dynamically conforms the thought that constant improvement can make things tend to perfect. The results also show that `optimal probability of false alarm and optimal power of finding out assignable cause(s) are same no matter how optimal sampling interval changes, optimal value of probability of false alarm is rather small and optimal power of finding out assignable cause(s) is as high as 1. It means there is no missing alarm. This result is very important because any missing alarm will cause the financial indices out of control and even bankruptcy of insurers. Finally, the results show that total cost increases with the increase of sampling interval. It means that the smaller the sampling interval, the smaller the control cost it is. Figure 4 indicates that there are two points are out of control region. It means that in last two years, the financial indices exist some problems and the insurer much take some effective measures to prevent this situation from further worse. Comparing to Mao (2019 a), the criteria of control is more strict in the situation of statistical-economic design than those of statistical design of multivariate special triangle control chart. That is, both of the optimal probability of false alarm and the power of finding out assignable cause(s) based on statistical-economic criteria are greater than those based on statistical design. In this way, it can be favorable for the manager to finding out assignable cause(s) as soon as possible. In addition, if the process is (are) affected by other assignable cause (s) and all or some parameters have changed, it is necessary to re-calculate optimal solutions and reset the optimal control region of the control chart.

Table 1 indicates that all of the six indices selected are growth indicators under specific ranges. However, Figure 1 and Figure 2 show that the change tarences of five indices, x_1, x_2, x_3, x_4 and x_6 gradually decline with time and only one index x_5 slightly increases with time. Therefore, based on the data of Table 1, we estimate the values of the slope vector of equation (1) by using linear regression method as listed in Table 2. Table 2, Figure 1 and Figure 2 have shown obvious decline trends of five financial indices, However, the points in Figure 4 has shown obviously gradually increases trends and there are two points out of control. The manager must not only pay much attention on the out of control situation resulting from the gradually increases of co-variances, and the interaction of multi-financial indices and co-variances, but also concern closely on the decline trends of these five indices and take effective measure to let the five indices gradually increases rather than continuously deceases. Although we cannot quantitatively determine the accurate adjustment value and interval of the vector of target values of financial indices due to that we cannot obtain relevant data, we can observe the change trends of financial indices by combining Table2, Figure 1, Figure 2 and Figure 4 and make effort to improve the soundness of insurers to avoid the situation become worse. It is important to notice that although the sixth indices is a growth index and it has been negative, it is necessary to carefully analysis the causes resulting in this worse situation and take some measures to make it become positive value.

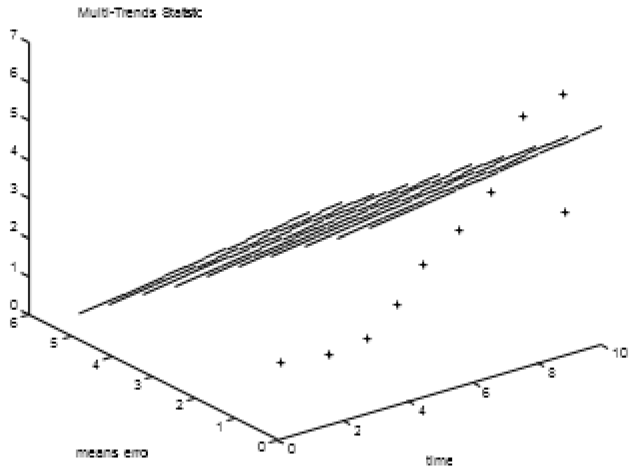

Figure 4. Multivariate Special Triangle Control Chart

Table 3. Optimal solutions of statistical-economics design of special triangle control chart.

$t=1\sim10$					
$h(t)$	0.11*	0.21	0.31	0.41	0.51
$ATS(t)$	0.11	0.21	0.31	0.41	0.51
$C^*(t)$	9.4826	9.4967	9.5111	9.5255	9.5402
$q(t)^*$	0.0163	0.0163	0.0163	0.0163	0.0163
$LV(t)^*$	5.6	5.6	5.6	5.6	5.6
$P^*(t)$	1	1	1	1	1
$n^*(t)$	5	5	5	5	5

Table 4. Sensitivity Analysis

$t=1\sim10$	$h^*(t)$	$ATS^*(t)$	$C^*(t)$	$q^*(t)$	$LV^*(t)$	$P^*(t)$	$n^*(t)$
$g(1+20\%)$	0.11	0.11	9.5929	0.0163	5.6	1	5
$g(1\text{-}20\%)$	0.11	0.11	9.3644	0.0103	4.667	1	6
$D(1+20\%)$	0.11	0.11	10.9503	0.0103	4.667	1	6
$D(1-20\%)$	0.11	0.11	8.0098	0.0163	5.6	1	5

$a_1(1+20\%)$	0.11	0.11	10.7298	0.0103	4.667	1	6
$a_1(1-20\%)$	0.11	0.11	8.2303	0.0163	5.6	1	5
$a_2(1+20\%)$	0.11	0.11	9.7920	0.0163	5.6	1	5
$a_2(1-20\%)$	0.11	0.11	9.1732	0.0163	5.6	1	5
$A_0(1+20\%)$	0.11	0.11	9.8174	0.0163	5.6	1	5
$A_0(1-20\%)$	0.11	0.11	9.1399	0.0103	4.667	1	6

Table 4 lists the results of sensitivity analysis when the important cost parameters change. The results in Table 4 show that optimal sampling interval and the power of finding out assignable cause(s) did not change no matter how the cost parameters change. The optimal total control cost, sample size and the control region changes with the change of the values of cost parameters. However, in one situation when the cost parameter of a_2 changes, only total optimal control cost changes and other optimal solutions do not change at all. On the whole, in all situations, the smaller the sampling interval and optimal average time of finding out assignable cause(s), the better they are. Therefore, collecting data and make calculation, drawing figures to carry out analysis in time is very important for the effective control of financial indices. This is different from the optimization and control when quality characteristics of production process shifting with time, where the sampling and inspection cost often high, especially when inspection of products needs high cost equipment or destructive test. Optimization of sampling interval needs to balance the sampling and inspection cost and the cost of out of control to determine an optimal sampling interval and optimal average time of finding out assignable cause (s).

Regarding to adjusting the financial indices of a firm when they gradually change with time, we should distinguish from the adjustment of the means of the production process. For the situation of monitoring financial indices, it is difficult to optimize the adjustment interval by quantitative methods because the financial indices generally have no serious specifications, therefore, it is difficult to evaluate the loss resulting from the unexpected shift of the financial indices and it is even impossible to estimate the adjustment cost. As a result, what is important is to distinguish two situations: on the one hand, if the change of the financial indices of a firm results from some seasonal or some other cyclical causes, the firm should make some off-seasonable or countercyclical measure to make the financial indices affected return to the normal states, on the other hand, if the change of the financial indices results from the gradual development and growth of the firms and in fact, these changes are sound changes expected by the firm. The important thing in this situation is to observe the change trends of each financial index and reset new and better target value vector as long as the original vector of target values is over the presetting maximum (minimum) level of the vector of process means.

We have discuss how to quantitatively determine the optimal adjustment interval for the production process in Mao (2019,a) Here we will not repeat it again.

CONCLUSION

In this paper, we discuss the statistical-economic design of multivariate special triangle control chart. We consider the optimal control multi-financial indices of a firm by using multi-objective programming. We illustrate its application in the optimal control of financial indices of insurers in U.S. Optimal solutions of control region, sampling interval, sample size, the probability of false alarming and the power of finding assignable cause(s) are determined. The optimal solutions satisfies the constraints of (1) power of finding out assignable cause(s) being greater or equal to a constant and at the same time; (2) the average time needed to detect out of control being equal or less than another constant. (3)the probability of false alarm being less than a not very small constant. Finally, we make some discussions on how to determine optimal interval of adjustment and make adjustment periodically for the optimal control of financial indices of a firm.

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Appendix 1

Prove LA_j and LB_j, $j=1,2,...,k$ are independent

Proof:

Assume $\varepsilon_{ij}, i=1,2,....k, iid$ and $\varepsilon_{ij} \sim N(0,\sigma_0^2)$

$$E\left(LA_j LB_j\right) = E\left(\sum_{i=1}^{n}(\boldsymbol{x_{ij}} - \boldsymbol{\hat{x}_{ij}})'\Sigma_0^{-1}\left(\boldsymbol{x_{ij}} - \boldsymbol{\hat{x}_{ij}}\right)\cdot\sum_{i=1}^{n}(\boldsymbol{\hat{x}_{ij}} - \boldsymbol{\bar{x}_j})'\Sigma_0^{-1}\left(\boldsymbol{\hat{x}_{ij}} - \boldsymbol{\bar{x}_j}\right)\right)$$

$$= E\left(\begin{array}{l}\sum_{i=1}^{n}\left(\boldsymbol{m}+\boldsymbol{b}t_j+\boldsymbol{\varepsilon_{ij}}-\boldsymbol{m}+\boldsymbol{b}t_j\right)\Sigma_0^{-1}\left(\boldsymbol{m}+\boldsymbol{b}t_j+\boldsymbol{\varepsilon_{ij}}-\boldsymbol{m}+\boldsymbol{b}t_j\right)\\ \cdot\sum_{i=1}^{n}\left((n-1)\boldsymbol{b}t_j+\boldsymbol{\varepsilon_{ij}}\right)'\Sigma_0^{-1}\left((n-1)\boldsymbol{b}t_j+\boldsymbol{\varepsilon_{ij}}\right)\end{array}\right)$$

$$= E\left(\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\,\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}}\cdot\sum_{i=1}^{n}\left((n-1)\boldsymbol{b}t_j+\boldsymbol{\varepsilon_{ij}}\right)'\Sigma_0^{-1}\left((n-1)\boldsymbol{b}t_j+\boldsymbol{\varepsilon_{ij}}\right)\right)$$

$$= E\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\,\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}}\sum_{i=1}^{n}\left((n-1)\boldsymbol{b}t_j\right)'\Sigma_0^{-1}\left((n-1)\boldsymbol{b}t_j\right)$$

$$+E\left(2(n-1)\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\,\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}}\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\,\Sigma_0^{-1}\boldsymbol{b}\sum_{i=1}^{n}t_j\right)+E\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\,\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}}\cdot\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\,\Sigma_0^{-1}\boldsymbol{\varepsilon}_{ij}$$

$$\overset{E\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}}=n}{\Rightarrow}\quad n\left(\sum_{i=1}^{n}\left(\left((n-1)\boldsymbol{b}t_j\right)'\Sigma_0^{-1}\left((n-1)\boldsymbol{b}t_j\right)\right)+n\right), \qquad \text{(A1)}$$

and $E(LA_j)E(LB_j)=n\left(\sum_{i=1}^{n}\left((n-1)\boldsymbol{bt}_j{}'\Sigma_0^{-1}\left((n-1)\boldsymbol{bt}_j\right)\right)+n\right),$ (A2)

Therefore, $E\left(LA_j\cdot LB_j\right)=E(LA_j)E(LB_j)$, (A3)

that is, LA_j and LB_j, $j=1,2,...,k$ are independent.

The following is the proof of $E\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}\,\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}} = n$.

Since $E\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}{}'\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}}=E\sum_{j=1}^{n}\sum_{k=1}^{p}\left(\varepsilon_{kij}\left(\sum_{s=1}^{p}\varepsilon_{kij}\sigma_{ks}^{-1}\right)\right)=E\sum_{h=1}^{n}\sum_{k=1}^{p}\varepsilon_{khj}\left(\sum_{s=1}^{p}\varepsilon_{khj}\sigma_{ks}^{-1}\right)$ (A4)

$E\left(\varepsilon_{khj}\sum_{i=1}^{n}\varepsilon_{kij}\right)=E\sum_{i=1}^{n}\varepsilon_{kij}=0$, when $h=i$, (A5)

$E\left(\varepsilon_{khj}\sum_{i=1}^{n}\varepsilon_{kij}\right)=E\varepsilon_{khj}E\sum_{i=1}^{n}\varepsilon_{kij}=0$, since ε_{khj} and ε_{kij} are iid when $h\neq i$, (A6)

and $E\sum_{i=1}^{n}\varepsilon_{kij}^{2}=n\sigma_k^2$, when $h=i$, (A7)

therefore, $E\sum_{i=1}^{n}\boldsymbol{\varepsilon_{ij}}{}'\Sigma_0^{-1}\boldsymbol{\varepsilon_{ij}}=n$, (A8)

where $\sigma_{ks}^{-1}, k=1,2,\text{L}\ ,p, s=1,2,\text{L}\ ,p$ are the elements of inverse metrix of covariance.

Appendix 2 (Please also see Mao (2019 (a))

x_1 - Ratio of total equity to total assets. The equity (or capital and surplus) provides a buffer in the event that losses are larger than expected (or investment returns are lower than expected). Capital also serves to protect the insured against reverse moral hazard by assuring the insured that the firm is committed to maintaining solvency without imposing costs on the insured through a unilateral decision to increase risk. Thus, higher value of this ratio indicates lower likelihood of insolvency

x_2 - Growth rate of net premium written. It is the index used in Insurance Regulatory information System (IRIS) created by the National Association of Insurance Commissioners (NAIC). High growth rate of new business in an insurance firm increases risk, because it requires higher levels of acquisition expenses (e.g., commissions, advertising), and it must be balanced by an appropriate level of capital and by risk management techniques. On the other hand, too low of a growth rate indicates that a firm is not competitive, and thus may also end up being insolvent if it is abandoned by its customers

x_3 - Net yield of investment assets. This index provides an indication of the quality of the insurer's investment portfolio. If the yield earned on investments is too low the insurer will show lower profitability, which by itself creates an insolvency risk, but it also lowers the chances of retaining existing customers and acquiring new ones, magnifying the risk. On the other hand, unusually high yield may also be problematic, as it may be an indicator of a high-risk investment portfolio.

x_4 - Ratio of net premium written to average equity. Property-casualty insurers commonly use the relationship of premium written to surplus (typically less than 3 as a benchmark of how much business they **should write.

x_5 - Retention ratio (NPW/GPW) indicates the portion of the business written that is not reinsured.

x_6 - Ratio of one-year reserve change to equity. This shows how much the firm's liability has changed in relation to surplus. A big change in liability without an appropriate change in surplus is an indicator of financial problems